

1. For  $\alpha = 4$  and for  $\alpha = 100$  solve the problem presented in class:

$$u'' - \alpha u = 4t^2 - 2$$
$$u'(0) = 0, u(1) = 0$$

- a) analytically
- b) using finite differences with  $h = 0.1$
- c) using 2 pt. Collocation
- d) single shooting

Legendre roots for collocation are given in the handout for  $-1 \leq z \leq 1$  (and appended below). For this problem, we need to scale these roots between zero and one, so the collocation points can be found from:  $t_i = 1/2(1 + z_i)$ .

2. Apply 5 point global collocation to the problem below and solve:

$$1/r^2 \partial/\partial r(r^2 \partial c/\partial r) = (c(r))^2$$

$$\partial c/\partial r = 0 \text{ at } r = 0$$

$$\partial c/\partial r = 100(c(1) - 1) \text{ at } r = 1$$

3. Solve problem 3 using finite differences with  $h = 0.1$  and  $h = 0.01$ .

4. Solve problem 2 using COLSYS, COLNEW or COLDAE with a small number of collocation points (say, 3) and a small number of finite elements (say 5-10). To get these codes, go to <http://www.netlib.org>. The handout shows a demo of COLDAE with the FORTRAN subroutines to solve problem 1. This can be modified to solve problem 2. Subroutines for COLSYS or COLNEW are similar.

Legendre-Gauss quadrature is a numerical integration method also called "the" Gaussian quadrature or Legendre quadrature. A Gaussian quadrature over the interval  $[-1, 1]$  with weighting function  $W(x) = 1$ . The abscissas for quadrature order  $n$  are given by the roots of the Legendre polynomials  $P_n(x)$ , which occur symmetrically about 0. The weights are

$$w_i = -\frac{A_{n+1} \gamma_n}{A_n P'_n(x_i) P_{n+1}(x_i)} \quad (1)$$

$$= \frac{A_n}{A_{n-1}} \frac{\gamma_{n-1}}{P_{n-1}(x_i) P'_n(x_i)}, \quad (2)$$

where  $A_n$  is the coefficient of  $x^n$  in  $P_n(x)$ . For Legendre polynomials,

$$A_n = \frac{(2n)!}{2^n (n!)^2} \quad (3)$$

(Hildebrand 1956, p. 323), so

$$\frac{A_{n+1}}{A_n} = \frac{[2(n+1)]!}{2^{n+1} [(n+1)!]^2} \frac{2^n (n!)^2}{(2n)!} \quad (4)$$

$$= \frac{2n+1}{n+1}. \quad (5)$$

Additionally,

$$\gamma_n = \int_{-1}^1 [P_n(x)]^2 dx \quad (6)$$

$$= \frac{2}{2n+1} \quad (7)$$

(Hildebrand 1956, p. 324), so

$$w_i = -\frac{2}{(n+1) P_{n+1}(x_i) P'_n(x_i)} \quad (8)$$

$$= \frac{2}{n P_{n-1}(x_i) P'_n(x_i)}. \quad (9)$$

Using the recurrence relation

$$(1-x^2)P'_n(x) = -nxP_n(x) + nP_{n-1}(x) \quad (10)$$

$$= (n+1)xP_n(x) - (n+1)P_{n+1}(x) \quad (11)$$

$$(12)$$

(correcting Hildebrand 1956, p. 324) gives

$$w_i = \frac{2}{(1-x_i^2) [P'_n(x_i)]^2} \quad (13)$$

$$= \frac{2(1-x_i^2)}{(n+1)^2 [P_{n+1}(x_i)]^2} \quad (14)$$

(Hildebrand 1956, p. 324).

The weights  $w_i$  satisfy

$$\sum_{i=1}^n w_i = 2, \quad (15)$$

which follows from the identity

$$\sum_{v=1}^n \frac{1-x_v^2}{(n+1)^2 [P_{n+1}(x_v)]^2} = 1. \quad (16)$$

The error term is

$$E = \frac{2^{2n+1} (n!)^4}{(2n+1) [(2n)!]^3} f^{(2n)}(\xi). \quad (17)$$

Beyer (1987) gives a table of abscissas and weights up to  $n = 16$ , and Chandrasekhar (1960) up to  $n = 8$  for  $n$  even.

$n$	$x_i$	$w_i$
2	$\pm 0.57735$	1.000000
3	0	0.888889
	$\pm 0.774597$	0.555556
4	$\pm 0.339981$	0.652145
	$\pm 0.861136$	0.347855
5	0	0.568889
	$\pm 0.538469$	0.478829
	$\pm 0.90618$	0.236927

The exact abscissas are given in the table below.

$n$	$x_i$	$w_i$
2	$\pm \frac{1}{3} \sqrt{3}$	1
3	0	$\frac{8}{9}$
	$\pm \frac{1}{3} \sqrt{15}$	$\frac{5}{9}$
4	$\pm \frac{1}{35} \sqrt{525 - 70\sqrt{30}}$	$\frac{1}{36} (18 + \sqrt{30})$
	$\pm \frac{1}{35} \sqrt{525 + 70\sqrt{30}}$	$\frac{1}{36} (18 - \sqrt{30})$
5	0	$\frac{128}{225}$
	$\pm \frac{1}{21} \sqrt{245 - 14\sqrt{70}}$	$\frac{1}{900} (322 + 13\sqrt{70})$
	$\pm \frac{1}{21} \sqrt{245 + 14\sqrt{70}}$	$\frac{1}{900} (322 - 13\sqrt{70})$

The abscissas for order  $n$  quadrature are roots of the Legendre polynomial  $P_n(x)$  meaning they are algebraic numbers of degrees 1, 2, 2, 4, 4, 6, 6, 8, 8, 10, 10, 12, ..., which is equal to  $2 \lfloor n/2 \rfloor$  for  $n > 1$  (Sloane's A052928).

Similarly, the weights for order  $n$  quadrature can be expressed as the roots of polynomials of degree 1, 1, 1, 2, 2, 3, 3, 4, 4, 5, 5, ..., which is equal to  $\lfloor n/2 \rfloor$  for  $n > 1$  (Sloane's A008619). The triangle of polynomials whose roots determine the weights is

$$\begin{aligned} x - 2 & (18) \\ x - 1 & (19) \\ 9x - 5 & (20) \\ 216x^2 - 216x + 49 & (21) \\ 45000x^2 - 32200x + 5103 & (22) \\ 2025000x^3 - 2025000x^2 + 629325x - 58564 & (23) \\ 1942943535000x^3 - 113071253400x^2 + 27510743799x - 1976763932 & (24) \\ 1707698764800000x^4 - 1707698764800000x^3 + \\ 606530263046400x^2 - 88878097916608x + 4373849390625 & (25) \end{aligned}$$

(Sloane's A112734).

#### REFERENCES:

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