06-606 **Computational Methods** **Fall, 2014**

Due: 11/5/14

Homework Assignment 6

1. While searching for the minimum of

$$f(x) = [x_1^2 + (x_2 + 1)^2][x_1^2 + (x_2 - 1)^2]$$

the algorithm terminates at the following points:

a) $x^{(1)} = [0,0]^T$
b) $x^{(2)} = [0,1]^T$
c) $x^{(3)} = [0,-1]^T$
d) $x^{(4)} = [1,1]^T$

Classify each point.

2. Consider the quadratic function with the parameter M:

$$f(x) = 3x_1 + x_2 + 2x_3 + 4x_1^2 + 3x_2^2 + 2x_3^2 + (M-2)\, x_1 x_2 + 2x_2 x_3$$

For M = 0 find all stationary points. Are they optimal? Find the path of optimal solutions as M increases from zero.

3. In Powell damping, the BFGS update is modified if $s^T y$ is not sufficiently positive by defining $\bar{y} = \theta y + (1-\theta) B^k s$ and substituting for y in the BFGS formula.

a) Show that θ can by found by solving the one-dimensional linear program:

$$\max \theta \; s.t. \theta s^T y + (1-\theta) s^T B^k s \geq 0.2\, s^T B^k s, \theta \in [0,1]$$

b) If $s^T y \geq 0.2\, s^T Bs$ show that Powell damping corresponds to a normal BFGS update.

c) If $s^T y \longrightarrow -\infty$, show that Powell damping corresponds to skipping the BFGS update.

4. Show that if B^k is positive definite $\cos\theta^k > 1/\kappa(B^k)$ where $\kappa(B^k)$ is the condition number of B^k, based on the 2-norm.

5. Derive a stepsize rule for α for the Armijo line search that minimizes the quadratic interpolant from the Armijo inequality.

6. Consider the convex problem:

$$\min x_1 \; s.t. x_2 \leq 0, x_2 - x_1^2 \geq 0$$

Show that this problem does not satisfy LICQ and does not satisfy the KKT conditions at its optimum solution.

7. Consider the convex problem

$$\min f(x) \;\; s.t.\, g(x) \leq 0$$

and the equivalent problem

$$\min f(x) \;\; s.t.\, g(x) + s = \; 0, s \geq 0.$$

a) Show that the KKT conditions of the two problems are equivalent.
b) If the second problem has a local solution. Show that this is also a global solution.

Solution Hw 6

1. Minimize

$$f(x) = (x_1^2 + (x_2+1)^2)(x_1^2 + (x_2-1)^2)$$
$$= x_1^4 + x_1^2(x_2^2 + 2x_2 + 1) + x_1^2(x_2^2 - 2x_2 + 1) + (x_2+1)^2(x_2-1)^2$$
$$= x_1^4 + x_1^2(2x_2^2 + 2) + (x_2^2 - 1)^2$$
$$= x_1^4 + 2x_1^2 x_2^2 + 2x_1^2 + x_2^4 - 2x_2^2 + 1$$

$$\nabla f(x) = \begin{bmatrix} 4x_1^3 + 4x_1 x_2^2 + 4x_1 \\ 4x_1^2 x_2 + 4x_2^3 - 4x_2 \end{bmatrix}$$

$$\nabla^2 f(x) = \begin{bmatrix} 12x_1^2 + 4x_2^2 + 4 & 8x_1 x_2 \\ 8x_1 x_2 & 4x_1^2 + 12x_2^2 - 4 \end{bmatrix}$$

a) $x_a^T = [0 \quad 0]$ $\qquad \nabla^2 f(x_a) = \begin{bmatrix} 4 & 0 \\ 0 & -4 \end{bmatrix}$

$\nabla f(x_a) = 0$

saddle point

b) $x_b^T = [0 \quad 1]$ $\qquad \nabla^2 f(x_b) = \begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix}$

$\nabla f(x_b) = 0$

local minimum - SSOC

c) $x_c^T = [0 \quad -1]$, $\nabla f(x_c) = 0$ $\quad \nabla^2 f(x_c) = \begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix}$

local min - SSOC

d) $x_d = [1, 1]^T$, $\nabla f(x_d) = \begin{bmatrix} 12 \\ 4 \end{bmatrix}$ not a stationary pt.

2. $f(x) = 3x_1 + x_2 + 2x_3 + 4x_1^2 + 3x_2^2 + 2x_3^2 + 2x_1x_2 + 2x_2x_3$

$$= \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}^T x + x^T \begin{bmatrix} 4 & (\frac{M}{2}-1) & \\ (\frac{M}{2}-1) & 3 & 1 \\ & 1 & 2 \end{bmatrix} x = a^T x + x^T A x$$

a) for $M = 0$

$f(x) = a^T x + x^T A x$ where $A = \begin{bmatrix} 4 & -1 & \\ -1 & 3 & 1 \\ & 1 & 2 \end{bmatrix}$

$2Ax + a = 0$

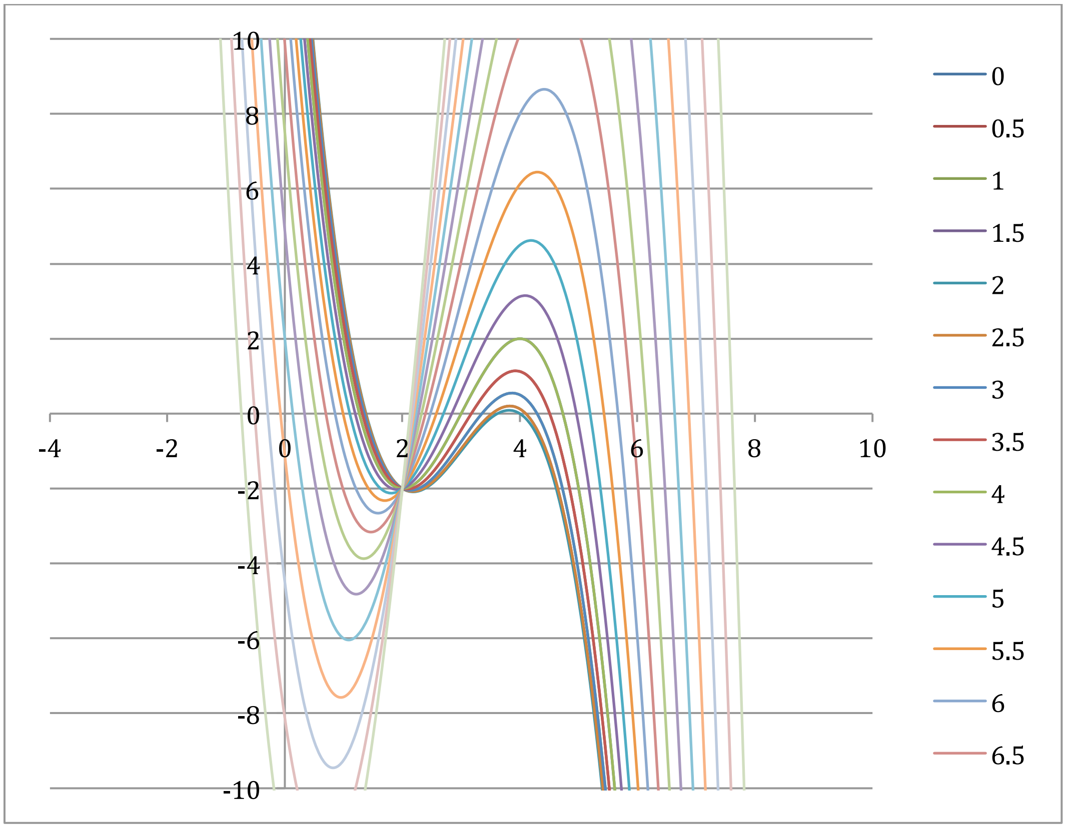
$x^* = \frac{1}{2} A^{-1} a$ gives the stationary points $(\nabla f(x^*) = 0)$

Eigenvalues of A determine whether $\nabla^2 f(x^*) = 2A$ is positive ~~semi-~~ definite or not.

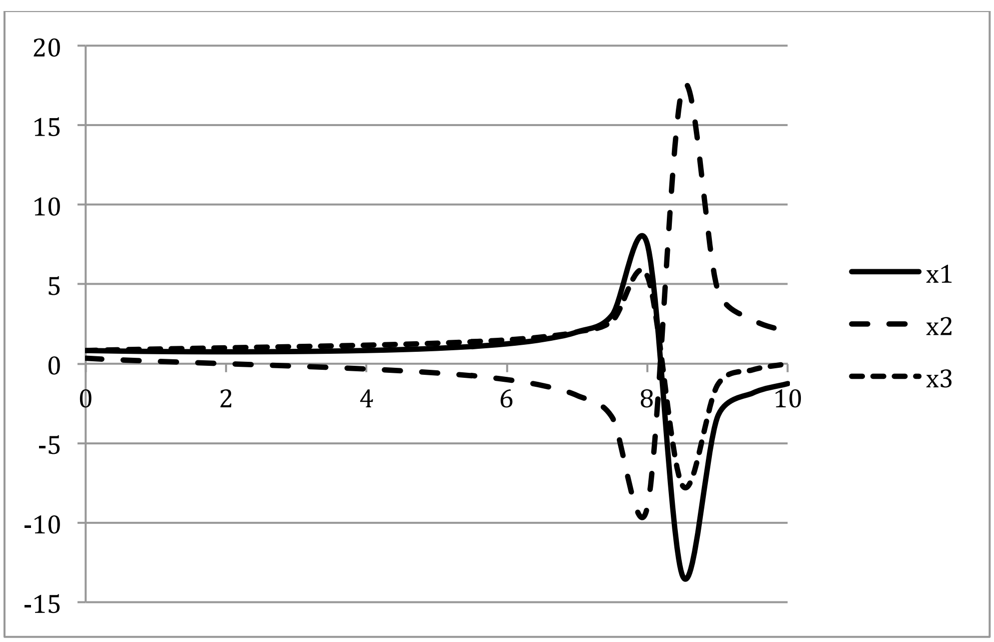
The attached curves show how x_1^*, x_2^*, x_3^* change with $M > 0$.

For $M > 8$, A is no longer positive definite and x^* is a saddle point. This is shown in the attached chart.

Roots of: $\det(A - \lambda I) = (4-\lambda)(3-\lambda)(2-\lambda) = (4-\lambda)(2-\lambda)(M/2 - 1)^2 = 0$ for M ε [0, 10].



Solution of $a^T x + x^T A x = 0$ as a function of M.



3. Powell damping:

$$\bar{y} = \theta y + (1-\theta) B^k s$$

so that $\bar{y}^T s \geq 0.2 s^T B^k s$ (n...

Using

$$B^{k+1} = B^k - \frac{B^k s s^T B^k}{s^T B^k s} + \frac{\bar{y}\bar{y}^T}{s^T \bar{y}}$$

$$s^T B^{k+1} s = s^T B^k s - \frac{(s^T B^k s)(s^T B^k s)}{s^T B^k s}$$

$$+ \frac{s^T \bar{y} \bar{y}^T s}{s^T \bar{y}} = s^T \bar{y}$$

$$= \theta(s^T y - s^T B^k s) + s^T B^k s$$

Now $\bar{y}^T s = \theta(s^T y - s^T B^k s) + s^T B^k s$

$$\geq 0.2 s^T B^k s$$

so $\theta \geq \dfrac{0.8 s^T B^k s}{s^T B^k s - s^T y}$

and $s^T B^{k+1} s \geq 0.2 s^T B^k s$

and B^{k+1} is p.d. along s.

a) Max θ

s.t. $\theta(y^T s - s^T B^k s) \geq -0.8 s^T B^k s$

$0 \leq \theta \leq 1$

$$-\theta - (y^T s - s^T B^k s) u + u_1 = 0$$

$$0 \leq 1 - \theta \perp u_1 \geq 0$$

$$0 \leq u \perp \theta(y^T s - s^T B^k s) + 0.8 s^T B s \geq 0$$

b) If $y^T s \geq 0.2\, s^T B s$ then

$$\theta(y^T s - s^T B s) > -0.8\, s^T B^k s$$

for any $\theta \in [0, 1]$ and $u = 0$ with $\theta = 1$, $u_1 = 1$.

If $y^T s < 0.2\, s^T B s$ then

$$\theta = \frac{0.8\, s^T B^k s}{s^T B^k s - s^T y}, \quad u_1 = 0 \quad \text{if } \theta > 0.$$

and $\bar{y}^T s = 0.2\, s^T B^k s$

c) If $s^T y \longrightarrow -\infty$ then $\theta = \dfrac{0.8\, s^T B^k s}{s^T B^k s - s^T y} \longrightarrow 0$

for $\theta = 0$ we have

$$s^T \bar{y} = s^T B s; \quad \bar{y} = B^k s$$

$$B^{k+1} = B^k - \frac{B^k s s^T B^k}{s^T B^k} + \frac{\bar{y} \bar{y}^T}{s^T \bar{y}}$$

$$= B^k$$

and no update is taken

4. B^k is positive definite

$$p = -(B^k)^{-1}\nabla f(x^k)$$

$$\cos\theta^k = \frac{-p^{k^T}\nabla f(x^k)}{\|p^k\|\,\|\nabla f(x^k)\|} = \frac{\nabla f(x^k)^T(B^k)^{-1}\nabla f(x^k)}{\|(B^k)^{-1}\nabla f(x^k)\|\,\|\nabla f(x^k)\|}$$

$$\geq \frac{\nabla f(x^k)^T(B^k)^{-1}\nabla f(x^k)}{\|\nabla f(x^k)\|^2\,\|(B^k)^{-1}\|} \qquad (1)$$

also with $(B^k)^{-1} = (L^T)^{-1}L^{-1}$

$$\nabla f(x^k)^T B^k\nabla f(x^k) = \|\nabla f(x^k)L^{-T}\|^2$$

and $\|\nabla f(x^k)\|^2 = \|\nabla f(x^k)L^{-T}B^kL^{-1}\nabla f(x^k)\|$

$$\leq \|\nabla f(x^k)L^{-T}\|^2\,\|B^k\|^2$$

so $\|\nabla f(x^k)L^{-T}\|^2 \geq \dfrac{\|\nabla f(x^k)\|^2}{\|B^k\|}$ (2)

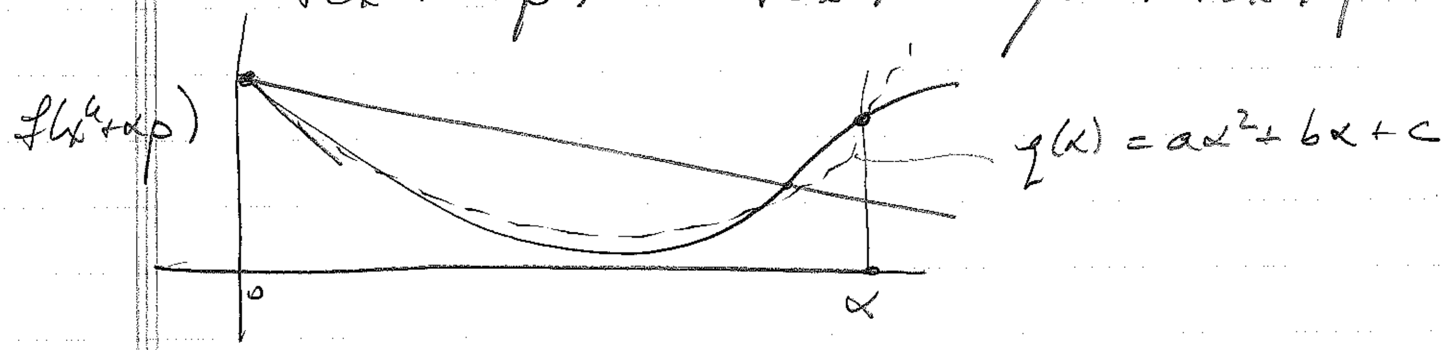
~~Substituting~~ (2) into (1) leads to:

$$\cos\theta^k \geq \frac{\|\nabla f(x^k)\|^2}{\|B^k\|\,\|\nabla f(x^k)\|^2\,\|(B^k)^{-1}\|}$$

$$= \frac{1}{\|B^k\|\,\|(B^k)^{-1}\|} = \frac{1}{\kappa(B^k)}$$

5. Armijo line search

$$f(x^k + \alpha p) \le f(x^k) + \eta \alpha \nabla f(x^k)^T p$$



$$f(x^k) = q(0) = c$$
$$f(x^k+\alpha p) = q(\alpha) = a\alpha^2 + b\alpha + c$$
$$\nabla f(x^k)^T p = q'(0) = b$$

$$f(x^k+\alpha p) = a\alpha^2 + (\nabla f(x^k)^T p)\alpha + f(x^k)$$
$$\longrightarrow \quad \left(f(x^k+\alpha p) - f(x^k) - (\nabla f(x^k)^T p)\alpha\right)/\alpha^2 = a$$

quadratic min is $q'(\bar\alpha) = 0 = 2a\bar\alpha + b$

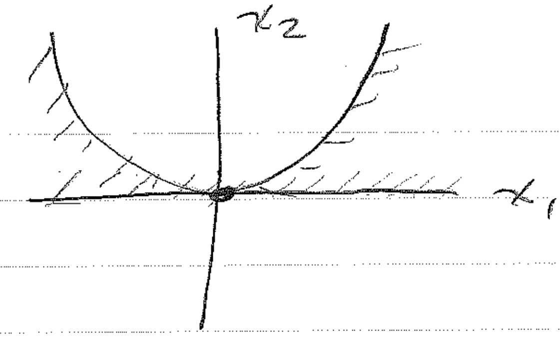
$$\bar\alpha = -b/2a = \tau\alpha$$
$$= \frac{\nabla f(x^k)^T p\, \alpha^2}{2\left(f(x^k+\alpha p) - f(x^k) - \alpha \nabla f(x^k)^T p\right)}$$

and $$\tau = \frac{\nabla f(x^k)^T p\, \alpha}{2\left(f(x^k+\alpha p) - f(x^k) - \alpha \nabla f(x^k)^T p\right)}$$

6. Min x_1

s.t. $x_2 \geq 0$

$x_2 \geq x_1^2$



$\mathcal{L} = x_1 - u_1 x_2 - u_2(x_2 - x_1^2)$

KKT conditions

$$1 + 2x_1 u_2 = 0$$
$$-u_1 - u_2 = 0$$

$x^* = 0$ so KKT conditions fail.

$$A^T = \begin{bmatrix} 0 & -1 \\ 2x_1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 0 & -1 \end{bmatrix}$$ @ x^* and LICQ fails.

7. Min $f(x)$ s.t $g(x) \leq 0$ (1)

(convex problem)

Min $f(x)$ s.t. $g(x) + s = 0$, $s \geq 0$ (2)

KKT conditions of (1)

$$\nabla f(x) + \nabla g(x) u = 0$$
$$0 \leq u \perp g(x) \leq 0$$

KKT conditions of (2)

$$\nabla f(x) + \nabla g(x) u = 0$$
$$u - \sigma = 0$$
$$0 \leq \sigma \perp s \geq 0$$

since $\sigma = u$, $s = -g(x)$ at solution and KKT conditions of (1) + (2) are identical since every solution of (2) is a solution of (1). + (1) has only global solutions, ~~then~~ (2) has only global solutions.