Advanced Process Systems Engineering - Exam I

06-720 2/27/08

This 120 minute exam consists of five unequally weighted parts (plus an extra credit problem). Please budget your time carefully. You may consult your class notes, textbooks and homework sets to solve these problems.

1. Kuhn Tucker Conditions

Consider the NLP problem:

$$\begin{aligned} &\text{Min } x_1 + (3x_2 - 1)^2 \\ \text{s.t. } &2x_1 + x_2 - x_3 = 0 \\ &x_1, x_2, x_3 \geq 0 \end{aligned}$$

a) Write the first order KKT conditions and find the solution and multipliers for this problem. (10 points)

b) At the solution of a), what is the reduced Hessian? Are second order KKT conditions satisfied? (10 points)

c) Given the solution in part a), classify each of the variables as basic, nonbasic and superbasic. (5 points)

2. Algebraic Equation Solving

a) For Marquardt's method, apply the equation for the search direction if an SVD factorization of J is used to solve the linear system. Simplify this equation as much as possible. (10 points)

b) Show that the search direction with λ sufficiently large is descent direction for $\frac{1}{2}(f^T f)$. (5 points)

3. Barrier Methods

a) Apply a barrier method to the problem Min $f(x)$, s.t. $c(x) = 0$, $a \leq x \leq b$. Show the primal-dual equations. (10 points)

b) Write the Newton step that corresponds to solving the primal-dual equations. (10 points)

5. Broyden's Method

Consider a system of equations with a linear subset, i.e., f(x) = 0, Ax = b. Show that with x^0 that satisfies $Ax^0 = b$ and an initial Broyden matrix of the form: $B^0 = \begin{bmatrix} C^0 \\ A \end{bmatrix}$, a Broyden matrix will always have the form: $B^k = \begin{bmatrix} C^k \\ A \end{bmatrix}$ and A will never be updated. (10 points)

6. SQP Methods

For the problem:

$$\begin{aligned} &Min\ \ 3x^2 + y^2 \\ &s.t.\ \ 3x + y = 6 \\ &xy = 0 \\ &x \geq 0,\ y \geq 0 \end{aligned}$$

a) For SQP, set up the QP step at iteration k for this problem by using W^k to represent the BFGS approximation to the Hessian matrix. (10 points)

b) For rSQP, set up the d_y (range-space) and d_z (null-space) steps at iteration k for this problem by using B^k to represent the BFGS approximation to the reduced Hessian matrix. (10 points)

c) Apply the KKT conditions directly to the problem, find the solution and the multipliers. Comment on the nature of this solution. (10 points)

Extra Credit: Convex Functions

Given an analytical function $f(x)$, prove that this function is convex if and only if its Hessian, $\nabla^2 f(x)$, is at least positive semi-definite for all x.

(Hint: use the property that $f(x) \geq f(x_1) + \nabla f(x_1)^T(x-x_1))$ (10 points)