06-720 Advanced Process Systems Engineering Spring, 2011

Homework 5 Due: 3/2/11

1. For the system $xy = 1, x \in [1/2, 2], y \in [1/2, 2]$:
   (a) Plot the McCormick relaxation for this problem.
   (b) If the region is partitioned into four with $x \leq 1, x \geq 1$ and $y \leq 1, y \geq 1$, plot the resulting regions.

2. Solve the problem $max\ x + y, s.t.\ xy \leq 4,\ x \in [0, 6],\ y \in [0, 4]$.

3. Consider the following NLP:

$$
\begin{aligned}
min \quad & x_1 - x_2 - x_3 - x_1x_3 + x_1x_4 + x_2x_3 - x_2x_4 \\
s.t. \quad & x_1 + 4x_2 \leq 8 \\
& 4x_1 + x_2 \leq 12 \\
& 3x_1 + 4x_2 \leq 12 \\
& 2x_3 + x_4 \leq 8 \\
& x_3 + 2x_4 \leq 8 \\
& x_3 + x_4 \leq 5 \\
& 0 \leq x_1, x_2, x_3, x_4 \leq 10
\end{aligned}
$$

   (a) Apply McCormick convex envelopes and develop the LP lower bounding problems. Solve the problem to a global solution.
   (b) Verify the solution to this problem by solving it with BARON.

4. Consider the integer programming problem:

$$
\begin{aligned}
max \quad & 1.2y_1 + y_2 \\
s.t. \quad & y_1 + y_2 \leq 1 \\
& 1.2y_1 + 0.5y_2 \leq 1 \\
& y_1, y_2 = \{0, 1\}
\end{aligned}
$$

   (a) Determine from inspection the solution of the relaxed problem.
   (b) Enumerate the four 0-1 combinations in your plot to find the optimal solution.
   (c) Solve the relaxed LP problem by hand and derive Gomory cuts based on the LP relaxation. Verify that they cut-off the relaxed LP solution.
   (d) Solve the above problem with the branch and bound method by enumerating the nodes in the tree and solving the LP subproblems with GAMS.

5. A company is considering to produce a chemical C which can be manufactured with either process II or process III, both of which use as raw material chemical B. B can be purchased from another company or else manufactured with process I which uses A as a raw material.

Consider the two following cases:
1. Maximum demand of C is 10 tons/hr with a selling price of \$1800/ton.
2. Maximum demand of C is 15 tons/hr; the selling price for the first 10 ton/hr is \$1800/ton, and \$1500/ton for the excess.

Investment and Operating Costs:

	Fixed (\$/hr)	Variable(\$/ton raw mat)
Process I	1000	250
Process II	1500	400
Process III	2000	550

Prices:
A: \$500/ton
B: \$950/ton

Conversions:

Process I: 90% of A to B
Process II: 82% of B to C
Process III: 95% of B to C

Maximum supply of A: 16 tons/hr

Given the specifications above, formulate an MILP model and solve it with GAMS to decide:

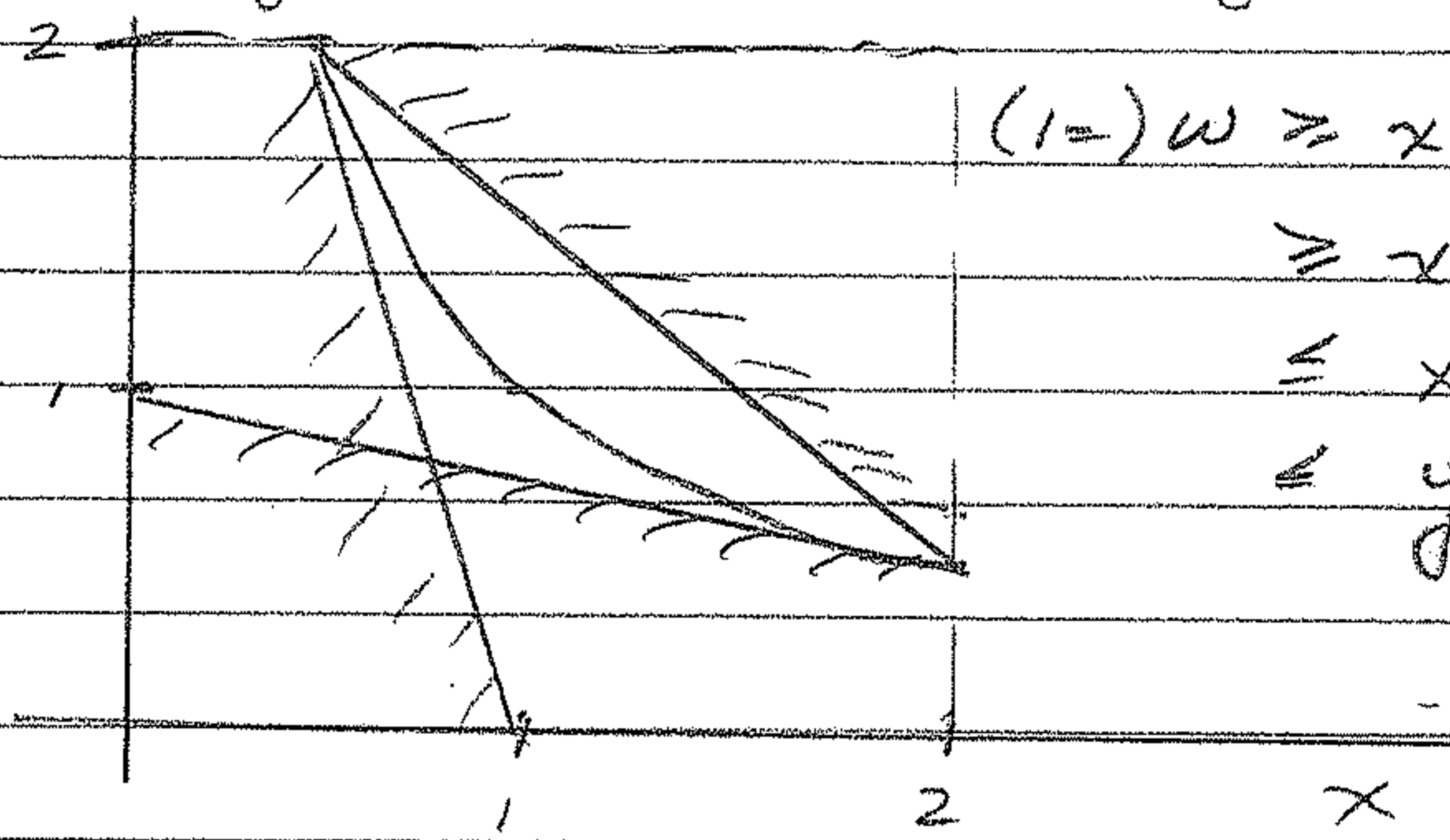
(a) Which process to build (II and III are exclusive)?

(b) How to obtain chemical B?

(c) How much should be produced of product C? The objective is to maximize profit.

Solution - Hwk 5

1) $xy = 1, \quad x \in [1/2, 2], \quad y \in [1/2, 2]$

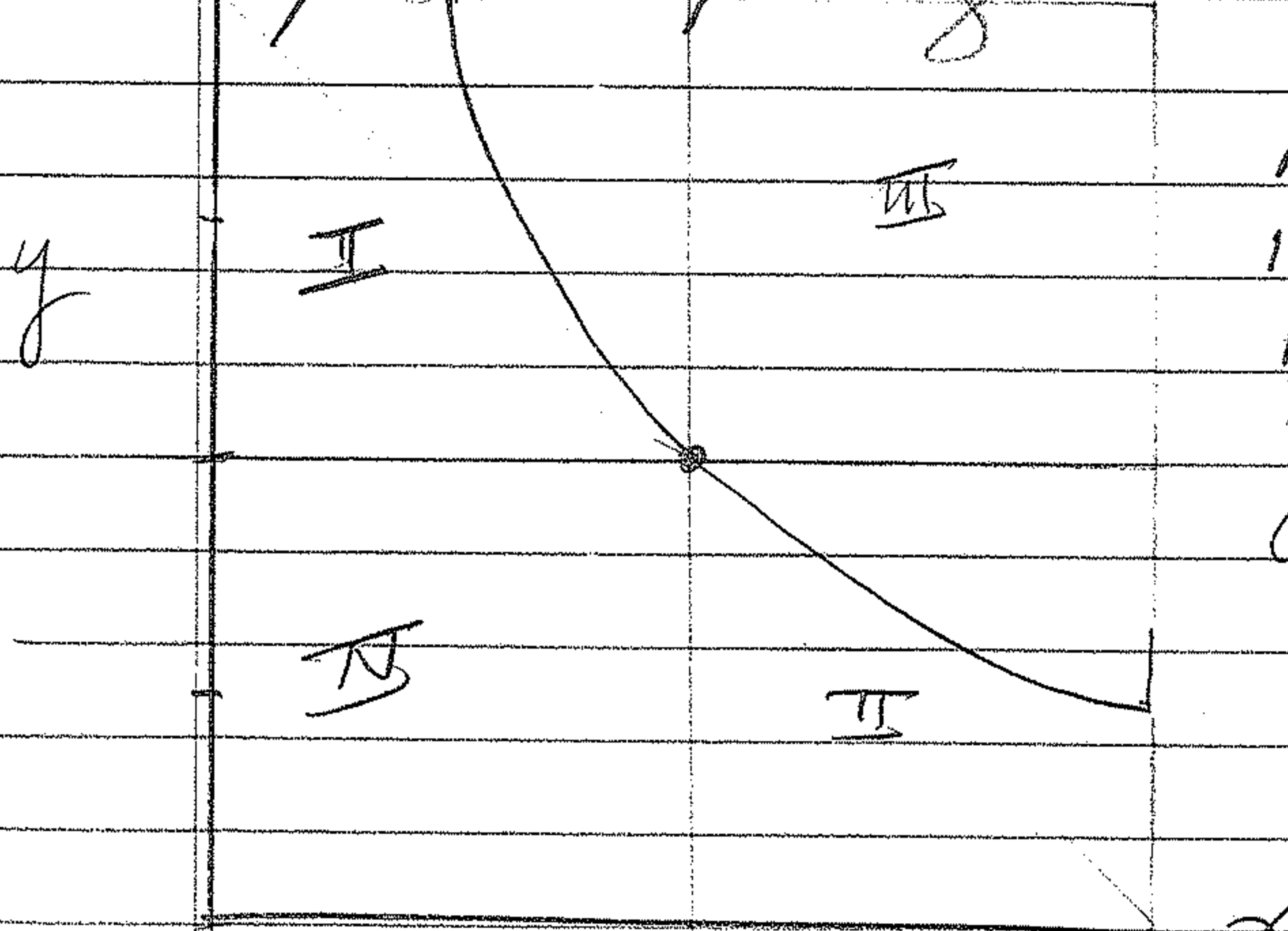


$$
\begin{aligned}
(1=)\, w &\geq x_L y + y_L x - x_L y_L \\
&\geq x_U y + y_U x - x_U y_U \\
&\leq x_U y + x y_L - x_U y_L \\
&\leq y_U x + x_L y - x_L y_U
\end{aligned}
$$

McCormick Relaxation:

$$
\begin{aligned}
1 &\geq y/2 + x/2 - 1/4 \\
1 &\geq 2x + 2y - 4 \\
1 &\leq 2y + x/2 - 1 \\
1 &\leq 2x + y/2 - 1
\end{aligned}
$$

Split into four regions



explore "empty regions"

$$
\begin{aligned}
1 &\geq y + x - 1 \\
1 &\geq 2y + 2x - 4 \\
1 &\leq 2y + x - 2 \\
1 &\leq 2x + y - 2
\end{aligned}
$$

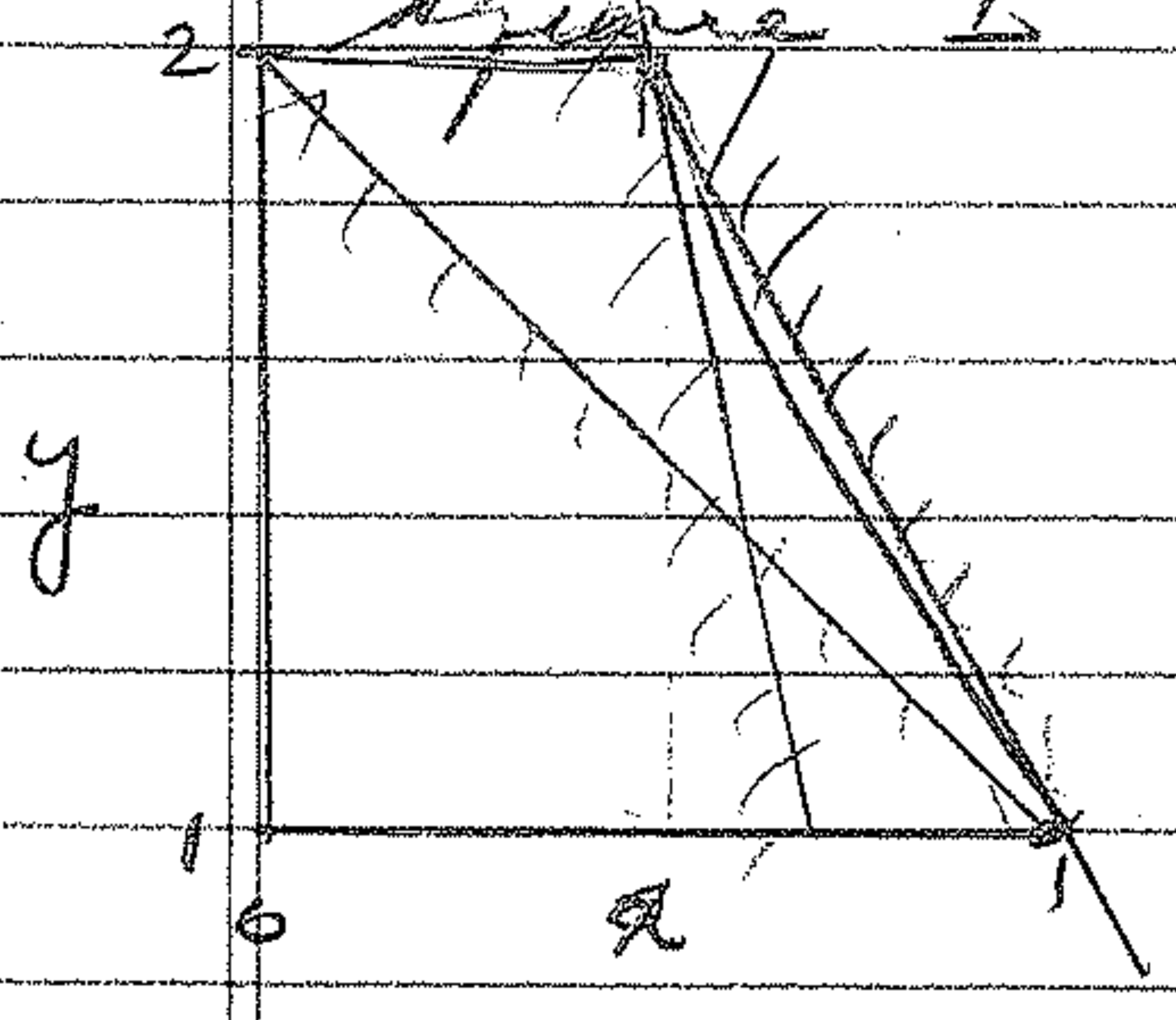
(outside square III)

$$
\begin{aligned}
1 &\geq 0 \\
1 &\geq y + x - 1 \\
1 &\leq y \\
1 &\leq x
\end{aligned}
$$

outside square IV

Consider squares I + II

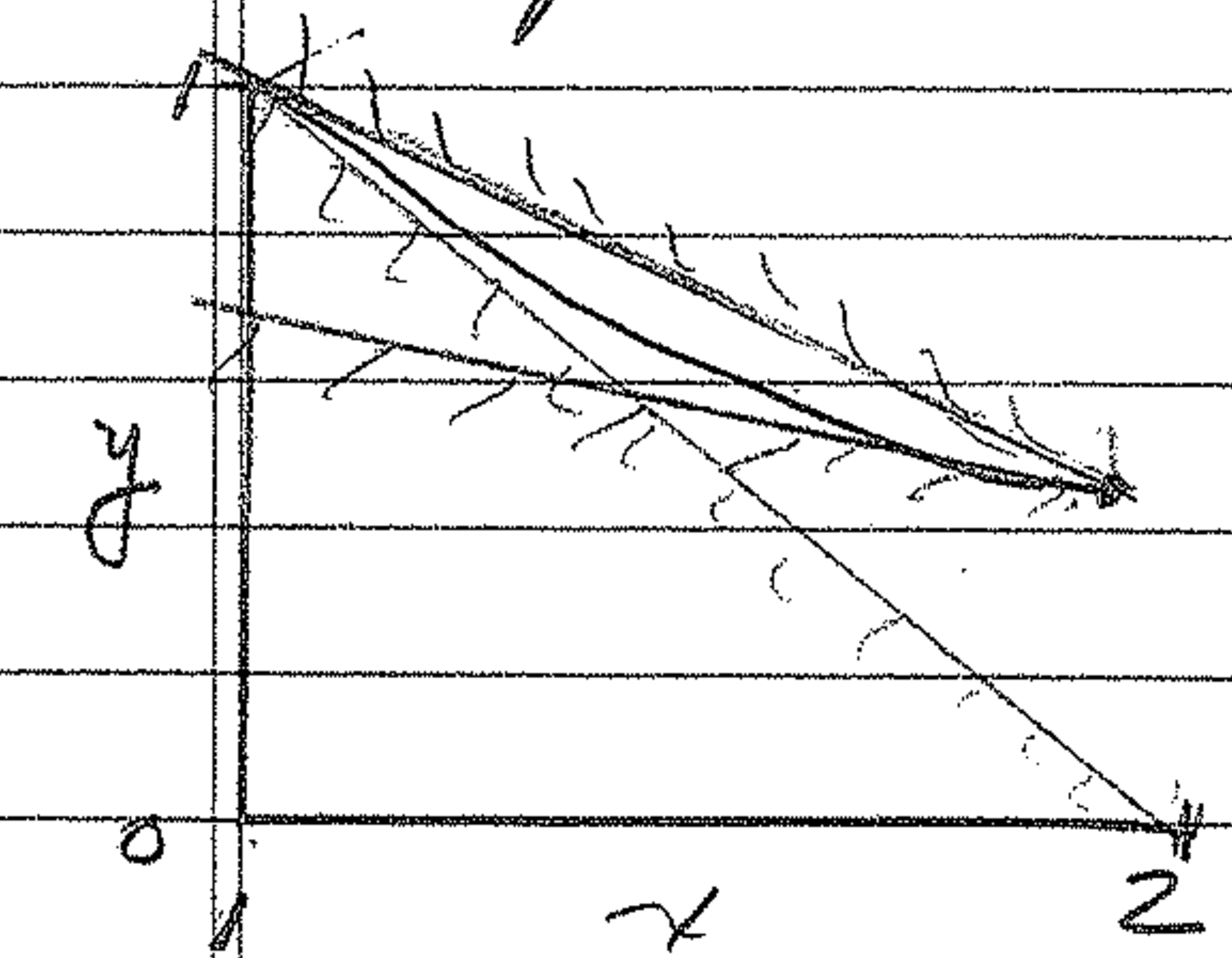
Square I



$x \in [1/2, 1]$, $y \in [1, 2]$

$$1 \geq y/2 + x - 1/2$$
$$1 \geq y + 2x - 2$$
$$1 \leq y + x - 1$$
$$1 \leq 2x + 1/2\, y - 1$$

Square II

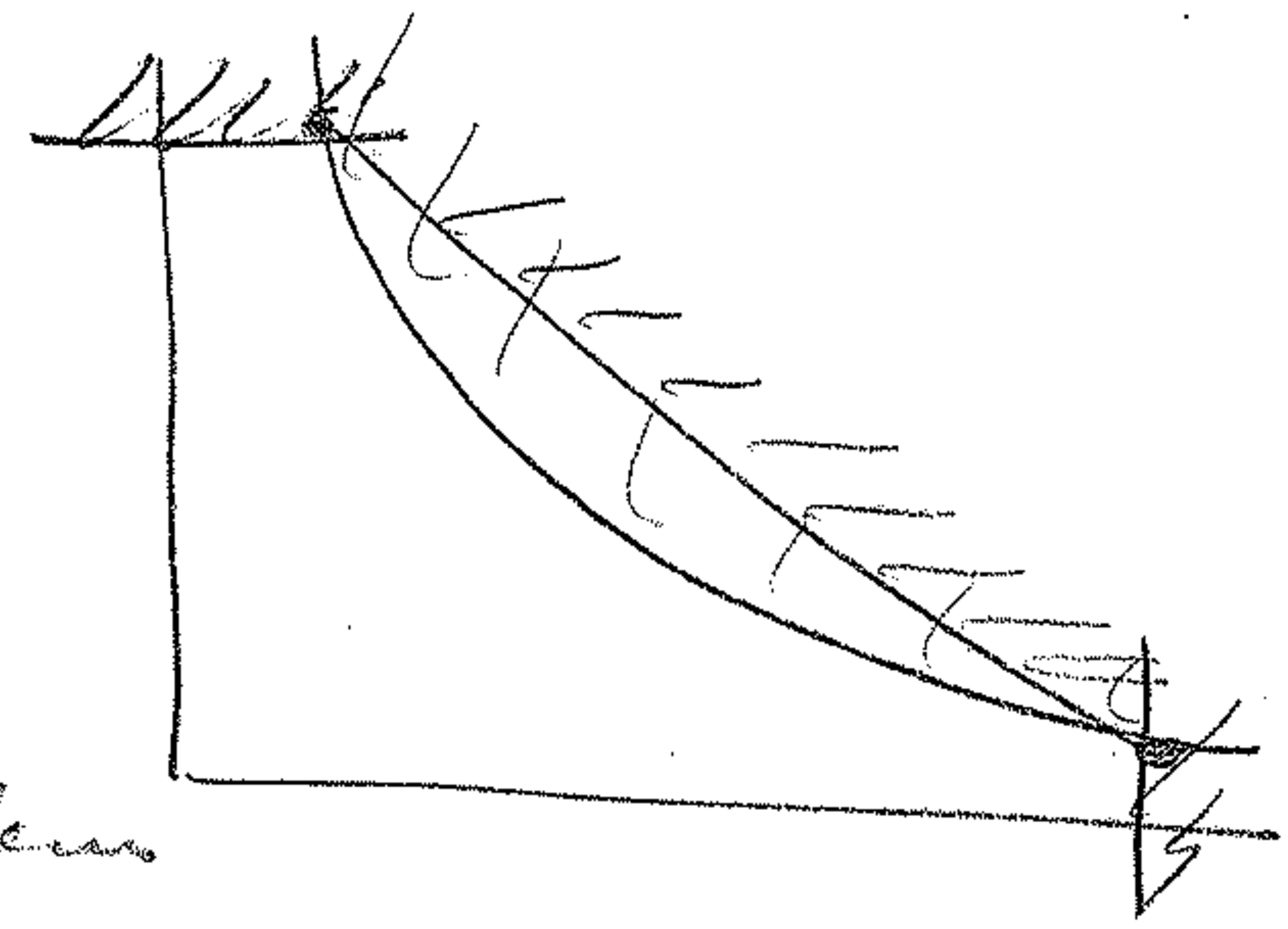


$x \in [1, 2]$, $y \in [1/2, 1]$

$$1 \geq x/2 + y - 1/2$$
$$1 \geq x + 2y - 2$$
$$1 \leq y + x - 1$$
$$1 \leq 2y + 1/2\, x - 1$$

Example - nonconvex region

(3)



$$\text{Min } -(x+y)$$
$$\text{s.t.} \quad xy \le 4$$
$$x \in [0, 6]$$
$$y \in [0, 4]$$

Lower bounding problem

$$\text{Min } -(x+y)$$
$$\text{s.t.} \quad w \le 4$$
$$w \geq xy^L + x^L y - x^L y^L$$
$$w \geq x y^U + y x^U - x^U y^U$$
$$w \le x y^L + x^U y - x^U y^L$$
$$w \le x y^U + x^L y - x^L y^U$$
$$0 \le x \le 6$$
$$0 \le y \le 4$$

for M_0 this becomes:

$$\text{Min } -(x+y)$$
$$w \le 4$$
$$w \geq 0$$
$$w \geq 6y + 4x - 24$$
$$w \le 6y$$
$$w \le 4x$$

Solution is: $x = 6, \ y = 2/3$

$$f_L = -6\tfrac{2}{3}$$
$$f_U = -6\tfrac{2}{3} \quad \text{(same point)}$$

3) Global Optimization

$$
\begin{aligned}
&Min \quad f = x_1 - x_2 - y_1 - x_1 y_1 + x_1 y_2 + x_2 y_1 - x_2 y_2 \\
&s.t. \quad x_1 + 2x_2 \le 8 && (1) \\
&\quad 4x_1 + x_2 \le 12 && (2) \\
&\quad 3x_1 + 4x_2 \le 12 && (3) \\
&\quad 2y_1 + y_2 \le 8 && (4) \\
&\quad y_1 + 2y_2 \le 8 && (5) \\
&\quad y_1 + y_2 \le 5 && (6) \\
&\quad 0 \le x_1, x_2, y_1 + y_2 \le 10 && (7)
\end{aligned}
$$

For lower bounds $x_1 = x_2 = y_1 = y_2 = 0$ satisfies all constraints.

For upper bounds,

$$\begin{aligned} &Max \quad x_1 \\ &s.t. \quad (1)-(7) \end{aligned} \quad \Rightarrow \quad x_1^U = 3$$

$$\begin{aligned} &Max \quad x_2 \\ &s.t. \quad (1)-(7) \end{aligned} \quad \Rightarrow \quad x_2^U = 2$$

$$\begin{aligned} &Max \quad y_1 \\ &s.t. \quad (1)-(7) \end{aligned} \quad \Rightarrow \quad y_1^U = 4$$

$$\begin{aligned} &Max \quad y_2 \\ &s.t. \quad (1)-(7) \end{aligned} \quad \Rightarrow \quad y_2^U = 4$$

Replace bilinear terms with other variables:

$$u_{11} = x_1 y_1 \qquad u_{12} = x_1 y_2 \qquad u_{21} = x_2 y_1 \qquad u_{22} = x_2 y_2$$

$$
\begin{aligned}
&u_{11} \le x_1 y_1^U && (8) \\
&u_{11} \le x_1^U y_1 && (9) \\
&u_{22} \le x_2 y_1^U && (10) \\
&u_{22} \le x_2^U y_2 && (11) \\
&u_{12} \ge 0 && (12) \\
&u_{12} \ge x_1^U y_2 + x_1 y_2^U - x_1^U y_2^U && (13) \\
&u_{21} \ge 0 && (14) \\
&u_{21} \ge x_2^U y_1 + x_2 y_1^U - x_2^U y_1^U && (15)
\end{aligned}
$$

We solve the final LP,

$$Min \quad f = x1 - x2 - y1 - u11 + u12 + u21 - u22$$

s.t.

(1) – (15)

The optimal solution is: $f^* = -13, x_1 = 3, x_2 = 0, y_1 = 4, y_2 = 0$
At the optimal solution, all variables are at their bounds, where McCormick convex envelopes are exact. Thus this is the global solution.

4) MILP Solutions

a)

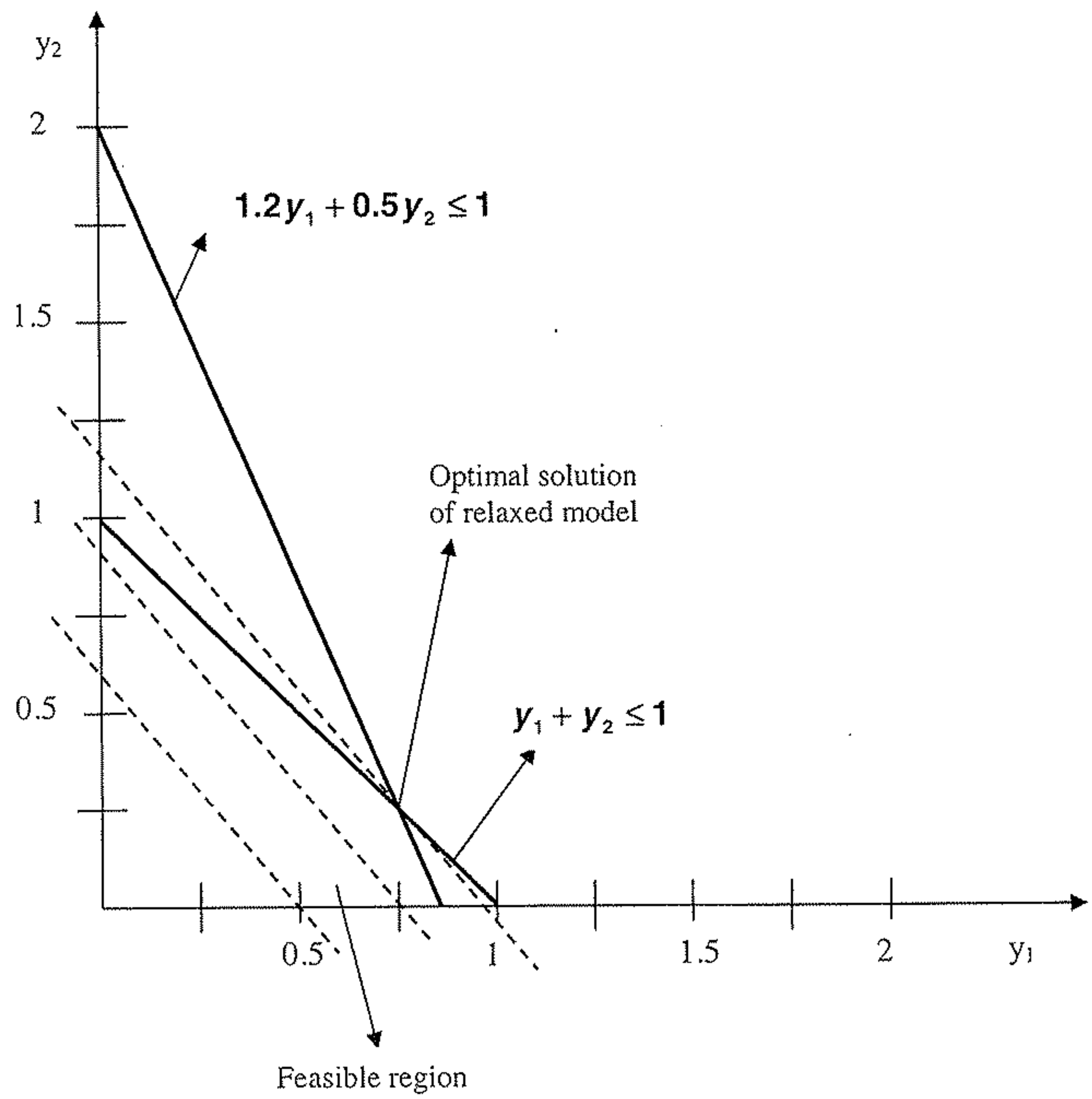
$$\textit{Max} \quad Z = 1.2y_1 + y_2$$

s.t.

$$y_1 + y_2 \le 1$$

$$1.2y_1 + 0.5y_2 \le 1$$

$$y_1, y_2 = 0,1$$



b)

By inspection, the optimum is the intersection of two lines: $y_1 + y_2 = 1$ and $1.2y_1 + 0.5y_2 = 1$ which is $y_1 = 5/7 = 0.72$ and $y_2 = 2/7 = 0.28$

b)

$y1 = 0 \quad y2 = 0 \quad Z = 0$

$y1 = 0 \quad y2 = 1 \quad Z = 1$

$y1 = 1 \quad y2 = 0 \quad \textit{Infeasible}$

$y1 = 1 \quad y2 = 1 \quad \textit{Infeasible}$

c)

The iterations in the simplex tableau is as follows:

	Z	Y1	Y2	S1	S2	RHS	Min Ratio
	1	1.2	1	0	0	0	
S1	0	1	1	1	0	1	1
S2	0	1.2	0.5	0	1	1	0.833333

	Z	Y1	Y2	S1	S2	RHS	
	1	0	0.5	0	-1	-1	
S1	0	0	0.583	1	-0.833	0.167	0.285
Y1	0	1	0.417	0	0.833	0.833	2

	Z	Y1	Y2	S1	S2	RHS	
	1	0	0	-0.857	-0.285	-1.142	all negative
Y2	0	0	1	1.714	-1.428	0.285	solution
Y1	0	1	0	-0.714	1.428	0.714	
		basic		nonbasic			

From the last two rows following Gomory cuts are generated,

$$y_2 + \lfloor 1.714 \rfloor s_1 + \lfloor -1.428 \rfloor s_2 \le \lfloor 0.285 \rfloor$$
$$y_2 + s_1 - 2s_2 \le 0$$

$$y_1 + \lfloor -0.714 \rfloor s_1 + \lfloor 1.428 \rfloor s_2 \le \lfloor 0.714 \rfloor$$
$$y_1 - s_1 + s_2 \le 0$$

At the optimal solution s1=0 and s2=0, so the Gomory cuts become, $y_2 \le 0$ and $y_1 \le 0$ which cuts the optimal solution of the relaxed problem $y_1 = 0.714$ and $y_2 = 0.285$.

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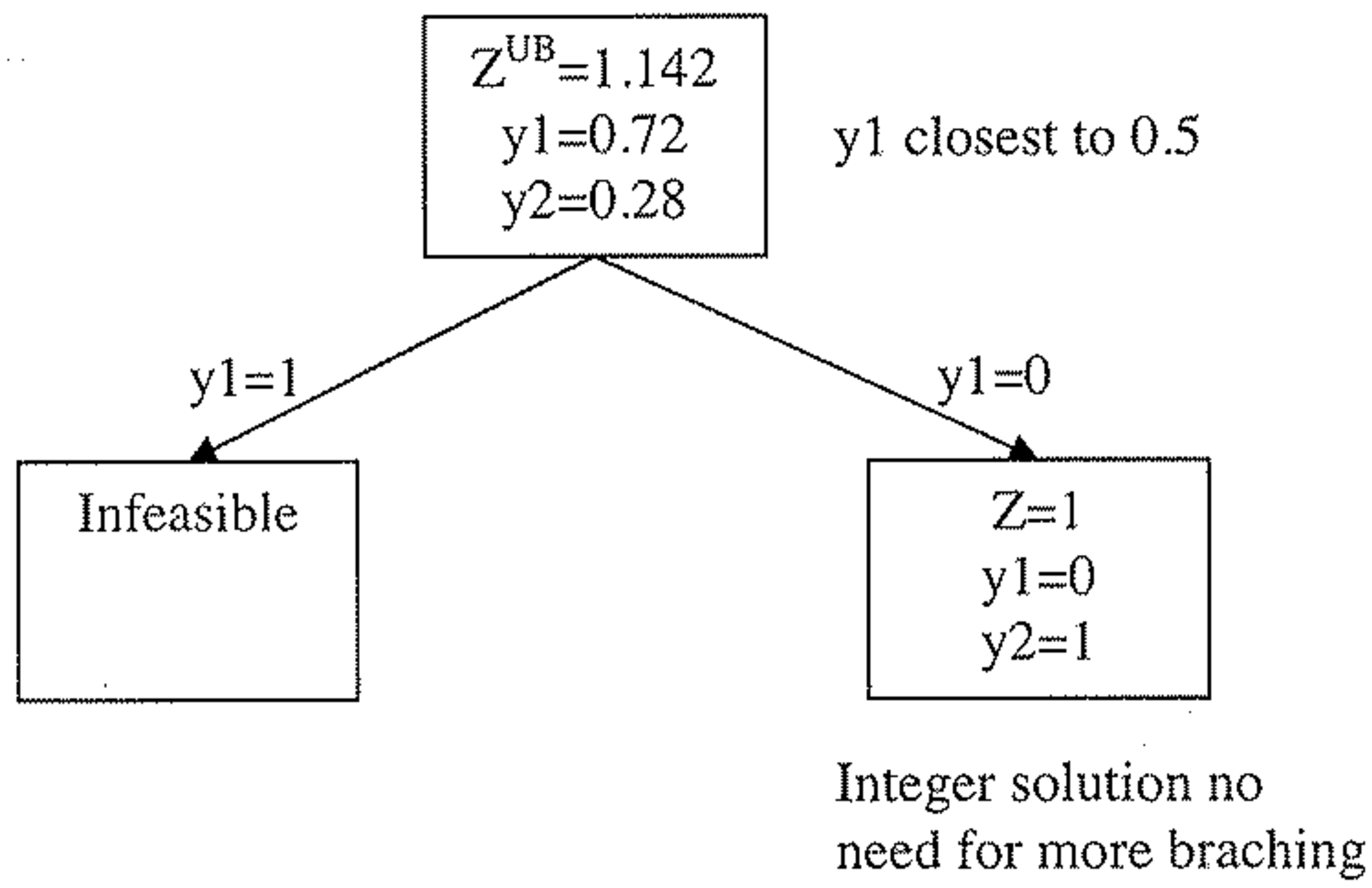
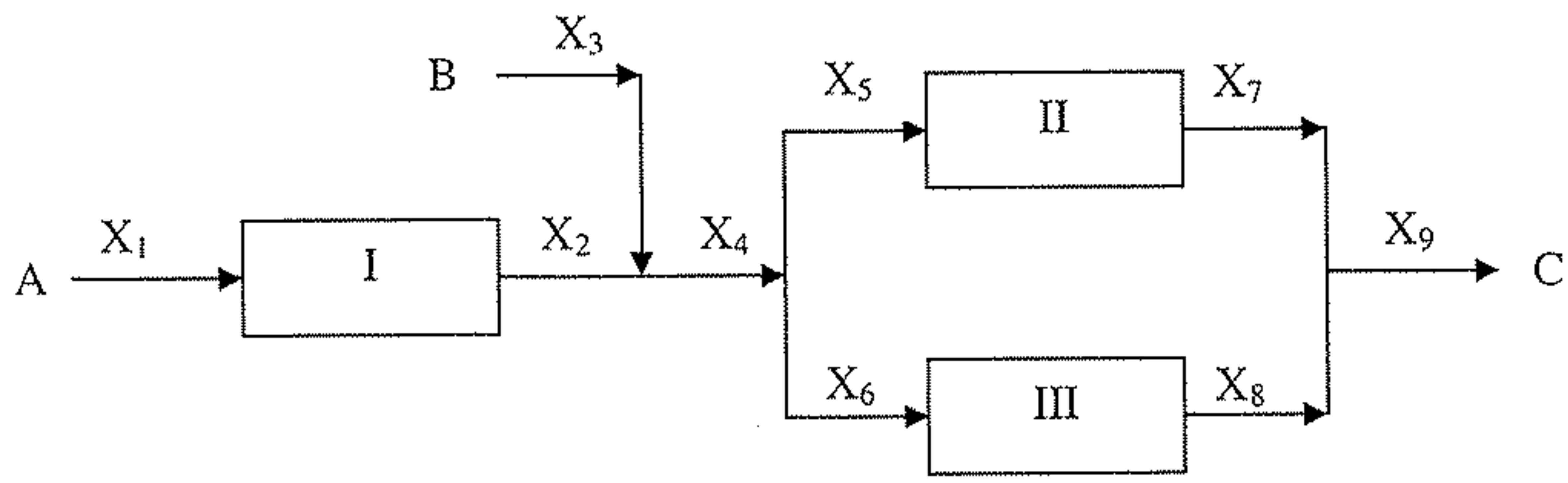
d)



Z^{UB}=1.142
y1=0.72
y2=0.28
y1 closest to 0.5
y1=1
y1=0
Infeasible
Z=1
y1=0
y2=1
Integer solution no need for more braching

5) Plant Case Study



Case I)

$$Max \quad Profit = -10y_1 - 15y_2 - 20y_3 + 18x_9 - (5+2.5)x_1 - 9.5x_3 - 4x_5 - 5.5x_6$$

s.t.

$$x_2 - 0.90x_1 = 0$$
$$x_7 - 0.82x_5 = 0$$
$$x_8 - 0.95x_6 = 0$$
$$x_4 - x_5 - x_6 = 0$$
$$x_9 - x_7 - x_8 = 0$$
$$x_4 - x_2 - x_3 = 0$$
$$x_1 - 16y_1 = 0$$
$$x_5 - 12.2y_2 = 0$$
$$x_6 - 10.6y_3 = 0$$
$$1 - y_1 + y_2 + y_3 \geq 1$$
$$x_1 \leq 16$$
$$x_9 \leq 10$$
$$x_i \geq 0 \qquad i = 1,\ldots,9$$
$$y_j = 0,1 \quad j = 1,2,3$$

Solution:

$$\text{Profit} = \$459.3/\text{hr}$$
$$x_1 = 13.55 \ tons/hr$$
$$x_2 = x_4 = x_5 = 12.20 \ tons/hr$$
$$x_7 = x_9 = 10 \ tons/hr$$
$$x_3 = x_6 = x_8 = 0 \ tons/hr$$

* Select processes I and II to produce chemical C.
* Chemical B is obtained from process I.
* Production rate of chemical C is 10 tons/hr.

Case II)
Partition x_9 into two:

$$x9 = \begin{bmatrix} x_9^1 \\ x_9^2 \end{bmatrix} \qquad \begin{matrix} c_9^1 = 18 & x_9^1 \le 10 \\ c_9^2 = 18 & x_9^2 \le 5 \end{matrix}$$

$$Max \quad Profit = -10y_1 - 15y_2 - 20y_3 + 18x_9^1 + 15x_9^2 - (5 + 2.5)x_1 - 9.5x_3 - 4x_5 - 5.5x_6$$

$$\begin{aligned}
&s.t.\\
&x_2 - 0.90x_1 = 0\\
&x_7 - 0.82x_5 = 0\\
&x_8 - 0.95x_6 = 0\\
&x_4 - x_5 - x_6 = 0\\
&x_9^1 + x_9^2 - x_7 - x_8 = 0\\
&x_4 - x_2 - x_3 = 0\\
&x_1 - 16y_1 \le 0\\
&x_5 - 18.3y_2 \le 0\\
&x_6 - 15.8y_3 \le 0\\
&1 - y_1 + y_2 + y_3 \ge 1\\
&x_1 \le 16\\
&x_9^1 \le 10\\
&x_9^2 \le 5\\
&x_i \ge 0 \qquad i = 1,\ldots,9\\
&y_j = 0,1 \quad j = 1,2,3
\end{aligned}$$

Solution:

$$\begin{aligned}
&\text{Profit} = \$600/\text{hr}\\
&x_1 = 16\ tons/hr\\
&x_2 = x_4 = x_6 = 14.4\ tons/hr\\
&x_8 = x_9 = 13.68\ tons/hr\\
&x_3 = x_5 = x_7 = 0\ tons/hr
\end{aligned}$$

* Select processes I and III to produce chemical C.
* Chemical B is obtained from process I.
* Production rate of chemical C is 13.68 tons/hr.